

# Sum Rules For Confining Potentials

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## Abstract

Using the Green's function associated with the one-dimensional Schroedinger equation it is possible to establish a hierarchy of sum rules involving the eigenvalues of confining potentials which have only a boundstate spectrum. For some potentials the sum rules could lead to divergences. It is shown that when this happens it is possible to examine the separate sum rules satisfied by the even and odd eigenstates of a symmetric confining potential and by subtraction cancel the divergences exactly and produce a new sum rule which is free of divergences. The procedure is illustrated by considering symmetric power law potentials and the use of several examples. One of the examples considered shows that the zeros of the Airy function and its derivative obey a sum rule and this sum rule is verified. It is also shown how the procedure may be generalised to establish sum rules for arbitrary symmetric confining potentials.

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# 1 Introduction

Green's functions  $G$  are used to study the solutions to inhomogeneous differential equations which satisfy homogeneous boundary conditions.  $G$  satisfies the same boundary conditions as the solution  $\Psi$  of a second order differential equation.  $G$  satisfies a corresponding differential equation with a delta function source term (Morse and Feshbach (1953)). For one-dimensional second order differential equations there are two methods for constructing the Green's function. In the first method  $G$  is constructed as a sum involving the eigenfunctions and eigenvalues of the homogeneous differential equation. In the second method  $G$  is constructed from two solutions of the homogeneous differential equation satisfying appropriate boundary conditions. The existence of two alternative ways of constructing the Green's function leads to the establishment of certain equalities involving the eigenfunctions and eigenvalues of the differential operator. For studying the boundstates of quarks with definite angular momenta the radial Schroedinger equation with a confining potential has been used extensively. Such solutions for zero orbital angular momentum (*viz*) the s-states, can be viewed as the odd states of a symmetric confining potential in the infinite space  $-\infty \leq x \leq \infty$  (Quigg *et al* (1980)). Since confining potentials have no scattering states and have only boundstates the corresponding Green's functions lead to a hierarchy of sum rules which only involve the boundstate eigenvalues. Such sum rules can be powerful tools to establish the extent to which a set of numerically computed eigenvalues approach completeness. A leading order sum rule has been used recently by Mezincescu (2000) to shed light on the hypothesis advanced by Bender and Boettcher (2001) that a cubic oscillator with imaginary coupling strength has a completely real spectrum.

The sum rules involving all the boundstate eigenvalues do not always lead to convergent expressions and prove useful. When this happens it is still possible to extract something useful from the Green's functions. In this paper it is shown that it is possible to establish separate sum rules for the odd and even states in a symmetric potential by considering Green's functions satisfying different boundary conditions and that when divergences in the sums arise they may be of the same type for the odd and even states. If this is the case then it is possible to subtract the two divergent sum rules to establish a convergent sum rule involving both the odd and even states. The plan of this paper is as follows: the construction of the Green's function and the derivation of the sum rules for the inverses of the eigenvalues of

confining potentials are presented in Section 2. The sum rules for the even and odd states in symmetric confining potentials of the power law form are constructed in Section 3. The spectral zeta function for power law potentials has been studied by Voros (2000). In Section 4 several solvable examples are considered and the sum rules are verified. In Section 5 the case of an arbitrary symmetric confining potential is considered and general results are presented.

## 2 Green's Functions and Sum Rules

Consider a second order differential operator of the form

$$L(x) = -\frac{d^2}{dx^2} + V(x) \quad (1)$$

with a complete set of solutions  $\Psi_n$  of the eigenvalue equation

$$L\Psi_n = \lambda_n\Psi_n \quad (2)$$

for a choice of two boundary conditions  $B_1$  and  $B_2$  on  $\Psi$  at  $x_1$  and  $x_2$  respectively.  $B_1$  and  $B_2$  can be chosen to be homogeneous boundary conditions of the Neumann or Dirichlet kind, i.e., the eigenfunction or its derivative vanishes at  $x_1$  or  $x_2$ . The orthonormality of the eigenfunctions  $\Psi_n$  is given by

$$\int_{x_1}^{x_2} \Psi_m(x) \Psi_n(x) = \delta_{mn}. \quad (3)$$

A Green's function  $G$  satisfying the same boundary conditions  $B_1$  and  $B_2$  as  $\Psi_n$  at  $x_1$  and  $x_2$ , respectively, may be defined as a solution of

$$(L(x) - \lambda) G(x, y; \lambda) = \delta(x - y). \quad (4)$$

A representation of  $G$  in terms of the complete set of eigenfunctions and eigenvalues is discussed in standard textbooks on Green's functions. For a general  $V(x)$  the complete set of eigenfunctions include eigenstates for both discrete and continuous eigenvalues. However, if  $V(x)$  is a confining potential, then the complete set is made up of only discrete eigenvalues  $\lambda_n$  and  $G$  may be represented as a discrete sum given by

$$G(x, y; \lambda) = \sum_n \frac{\Psi_n(x) \Psi_n(y)}{\lambda_n - \lambda}. \quad (5)$$

Using the completeness relation

$$\sum_n \Psi_n(x) \Psi_n(y) = \delta(x - y) \quad (6)$$

and eq.(2) it is easy to verify that  $G$  defined by eq.(5) satisfies eq.(4).

For one-dimensional equations such as eq.(4) an alternative procedure to construct  $G$  is to solve eq.(4) directly. Let  $\phi_1$  and  $\phi_2$  be two linearly independent solutions of

$$(L - \lambda) \phi_{1,2} = 0 \quad (7)$$

chosen such that  $\phi_1$  satisfies the boundary condition  $B_1$  but not  $B_2$  and  $\phi_2$  satisfies  $B_2$  but not  $B_1$ . Then in terms of  $\phi_1$  and  $\phi_2$ ,  $G$  may be constructed as

$$G(x, y; \lambda) = -\frac{\phi_1(x_<, \lambda) \phi_2(x_>, \lambda)}{W(\phi_1, \phi_2)} \quad (8)$$

where the Wronskian  $W$  given by

$$W(\phi_1, \phi_2) = \phi_1(x, \lambda) \frac{d\phi_2(x, \lambda)}{dx} - \phi_2(x, \lambda) \frac{d\phi_1(x, \lambda)}{dx} \quad (9)$$

satisfies

$$\frac{dW}{dx} = 0 \quad (10)$$

and  $x_< (x_>)$  is the smaller (larger) of  $(x, y)$ . Using eqs.(7) and (10) it is easy to verify that  $G$  defined by eq.(8) is a solution of eq.(4). When the Green's function can be constructed from eq.(8) then eq.(8) may be taken to be the definition of  $G$  and eq.(5) may be viewed as the expression of an equality relating  $G$  to the eigenfunctions and eigenvalues. The existence of two expressions for  $G$  leads to certain equalities. Using the orthonormality condition defined by eq.(3) it can be shown from eq.(5) that

$$\int_{x_1}^{x_2} G(x, x; \lambda) dx = \sum_n \frac{1}{(\lambda_n - \lambda)}. \quad (11)$$

Eq.(11) is an expression of a sum rule if eq.(8) for  $G$  is used to evaluate the integral. The above procedure can be generalised to provide a hierarchy of sum rules as discussed by Sukumar (1990). The orthonormality of the eigenfunctions can be shown to lead to a second order sum rule in the form

$$\int_{x_1}^{x_2} dx \int_{x_1}^{x_2} dy G(x, y; \lambda) G(y, x; \lambda) = \sum_n \frac{1}{(\lambda_n - \lambda)^2}. \quad (12)$$

Similar sum rules involving a chain of  $n$  Green's functions starting from  $x$  and returning to  $x$  via  $(n - 1)$  intermediate set of points may be used to establish sum rules of arbitrary order  $n$ . The sums over the powers of the inverses of the eigenvalues for the case  $\lambda = 0$  are referred to as generalised zeta functions in the literature. All the sum rules defined so far converge for finite values of  $x_1$  and  $x_2$ . For general confining potentials with  $x_1 \rightarrow -\infty$  and  $x_2 \rightarrow \infty$  extra convergence conditions are needed to ensure that the sum rules lead to convergent expressions. These questions are examined in the next section.

### 3 Sum Rules for Power Law Potentials

Consider symmetric potentials of the power law form  $|x|^N$ . The eigenstates split into groups of odd states satisfying the boundary conditions

$$Lt_{x \rightarrow 0} \Psi(x) \rightarrow 0, Lt_{x \rightarrow \infty} \Psi(x) \rightarrow 0 \quad (13)$$

and even states satisfying the boundary conditions

$$Lt_{x \rightarrow 0} \frac{d}{dx} \Psi(x) \rightarrow 0, Lt_{x \rightarrow \infty} \Psi(x) \rightarrow 0. \quad (14)$$

If the parameter  $\lambda$  is chosen to be equal to 0 then the zero energy Schroedinger equation

$$\frac{d^2}{dx^2} \Psi = |x|^N \Psi \quad (15)$$

under the substitutions

$$\nu = \frac{2}{N+2}, z = \nu x^{\frac{1}{\nu}}, \beta = \frac{1}{N+2}, \Psi = z^\beta \Phi \quad (16)$$

transforms into

$$\left( \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} - \left( 1 + \frac{\beta^2}{z^2} \right) \right) \Phi = 0 \quad (17)$$

which is the differential equation satisfied by the modified Bessel functions (Abramowitz and Stegun (1965)). Hence the solutions are of the form

$$\Psi = \sqrt{x} I_{\pm\beta} \left( \nu x^{\frac{1}{\nu}} \right) \quad (18)$$

from which we can construct solutions which satisfy appropriate boundary conditions as  $x \rightarrow 0$  and  $x \rightarrow \infty$ . The solution which shows decaying behaviour as  $x \rightarrow \infty$  is  $\sqrt{x}K_\beta(z)$ , the solution which vanishes as  $x \rightarrow 0$  is  $\sqrt{x}I_{+\beta}(z)$  and the solution for which the derivative vanishes as  $x \rightarrow 0$  is  $\sqrt{x}I_{-\beta}(z)$ . Hence the Green's function appropriate for the description of eigenstates satisfying boundary conditions analogous to those in eq. (13) is

$$G_1(x, y) = \nu \sqrt{x} \sqrt{y} I_{+\beta} \left( \nu x^{\frac{1}{\nu}}_{<} \right) K_\beta \left( \nu x^{\frac{1}{\nu}}_{>} \right) \quad (19)$$

and the Green's function appropriate for the description of eigenstates satisfying boundary conditions analogous to those in eq.(14) is

$$G_2(x, y) = \nu \sqrt{x} \sqrt{y} I_{-\beta} \left( \nu x^{\frac{1}{\nu}}_{<} \right) K_\beta \left( \nu x^{\frac{1}{\nu}}_{>} \right) . \quad (20)$$

The boundary conditions satisfied by  $G_1$  and  $G_2$  are

$$\begin{aligned} Lt_{x \rightarrow 0} G_1(x, y) &\rightarrow 0, \quad Lt_{x \rightarrow \infty} G_1(x, y) \rightarrow 0 \\ Lt_{x \rightarrow 0} \frac{d}{dx} G_2(x, y) &\rightarrow 0, \quad Lt_{x \rightarrow \infty} G_2(x, y) \rightarrow 0. \end{aligned} \quad (21)$$

The WKB result for the eigenvalues  $\lambda_n$  in the limit of large quantum number  $n$  can be derived from the Bohr-Sommerfeld formula and is of the form

$$\lambda_n = \left( \left( n + \frac{1}{2} \right) \frac{\sqrt{\pi} (N+2) \Gamma \left( \frac{N+2}{2N} \right)}{2 \Gamma \left( \frac{1}{N} \right)} \right)^{\frac{2N}{N+2}} \quad (22)$$

which shows that the sums over the inverses of the eigenvalues will not converge if  $N < 2$ . This also indicates that the integrals of the Green's functions over the infinite domain  $[0, \infty]$  will not converge if  $N < 2$ . When the Green's functions in eqs.(19) and (20) are used in eq.(11) with the choice of limits  $x_1 = 0$  and  $x_2 = \infty$  the resulting integrals can be performed (Gradshteyn and Ryzhik (1965)) if  $N > 2$  (*i.e*)  $\beta < \frac{1}{4}$ . The resulting sum rules are

$$S_1 = \sum_{n=0}^{\infty} \frac{1}{\lambda_{2n+1}} = \beta^{2-4\beta} \frac{\Gamma(3\beta) \Gamma(2\beta) \Gamma(1-4\beta)}{\Gamma(1-2\beta) \Gamma(1-\beta)}, \quad \beta < \frac{1}{4} \quad (23)$$

for the odd states and

$$S_2 = \sum_{n=0}^{\infty} \frac{1}{\lambda_{2n}} = \beta^{2-4\beta} \frac{\Gamma(2\beta) \Gamma(\beta) \Gamma(1-4\beta)}{\Gamma(1-3\beta) \Gamma(1-2\beta)}, \quad \beta < \frac{1}{4} \quad (24)$$

for the even states. The WKB approximation for the eigenvalues given by eq.(22) indicates that when  $N \leq 2$  the sums given by  $S_1$  and  $S_2$  diverge but it also indicates that if we look at the difference  $S_2 - S_1$  the terms in the resultant sum have the large  $n$  behaviour of  $n^{-\frac{3N+2}{N+2}}$  which would lead to convergent sums for all positive definite values of  $N > 0$ . Hence we examine

$$S_2 - S_1 = \int_0^\infty (G_2(x, x) - G_1(x, x)) dx. \quad (25)$$

Using the addition formulae satisfied by the modified Bessel functions which appear in the integrand the integral can be written as

$$S = S_2 - S_1 = \frac{4\beta}{\pi} \sin \pi \beta \int_0^\infty x K_\beta^2 \left( \nu x^{\frac{1}{\nu}} \right) dx \quad (26)$$

and the resulting integral can be performed (Gradshteyn and Ryzhik (1965)) to obtain the result

$$S = \sum_{n=0}^{n=\infty} \frac{(-)^n}{\lambda_n} = \beta^{2-4\beta} \frac{\Gamma(3\beta) \Gamma^2(2\beta) \Gamma(\beta)}{\Gamma(4\beta)} \frac{\sin \pi \beta}{\pi}. \quad (27)$$

By using the reflection property satisfied by the gamma functions

$$\Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad (28)$$

the results for the sum rules can be simplified.  $S$  can be represented in the form

$$S = \sum_{n=0}^{\infty} \frac{(-)^n}{\lambda_n} = \beta^{2-4\beta} \frac{\Gamma(3\beta) \Gamma^2(2\beta)}{\Gamma(4\beta) \Gamma(1-\beta)}, \quad N > 0, \quad \beta < \frac{1}{2}. \quad (29)$$

In terms of  $S$  the other sum rules derived in this section may be written in the compact form

$$S_2 = \sum_{n=0}^{\infty} \frac{1}{\lambda_{2n}} = S \frac{\sin 3\pi\beta}{\sin \pi\beta} \frac{1}{2 \cos 2\pi\beta}, \quad N > 2, \quad \left( \beta < \frac{1}{4} \right), \quad (30)$$

$$S_1 = \sum_{n=0}^{\infty} \frac{1}{\lambda_{2n+1}} = S \frac{1}{2 \cos 2\pi\beta}, \quad N > 2, \quad \left( \beta < \frac{1}{4} \right). \quad (31)$$

We have thus provided a direct proof of the sum rules listed by Voros (2000) by a method that is capable of being generalised to other symmetric confining potentials. In the next section special values of the power law index  $N$  for which the eigenvalues are readily available in the literature are considered and the resulting sum rules are examined.

## 4 Examples of Sum Rules

### 4.1 Linear Potential $V(x)=x$

For the linear potential the Schroedinger equation for zero energy is given by eq.(15) with  $N = 1$  and can be identified as the differential equation satisfied by Airy functions (Abramowitz and Stegun (1965)). The eigenvalues of the even eigenstates whose derivatives vanish at  $x = 0$  are given by the negative of the zeros of the derivative of the Airy function and the eigenvalues of the odd eigenstates whose wavefunctions vanish at  $x = 0$  are given by the negative of the zeros of the Airy function. The zeros of the Airy function and its derivative are listed in Abramowitz and Stegun (1965) and the first 10 zeros are given below in Table 1.

$n$	$\lambda_{2n}$	$\lambda_{2n+1}$	$\lambda_{2n}^{-1} - \lambda_{2n+1}^{-1}$
0	1.01879	2.33811	0.55386
1	3.24820	4.08795	0.06324
2	4.82010	5.52056	0.02632
3	6.16331	6.78671	0.01490
4	7.37218	7.94413	0.00977
5	8.48849	9.02265	0.00697
6	9.53545	10.04017	0.00527
7	10.52766	11.00852	0.00415
8	11.47506	11.93602	0.00337
9	12.38479	12.82878	0.00279

Table 1: The eigenvalues of the lowest 10 even and odd eigenstates for the linear potential and the differences between the inverses of the eigenvalues taken in pairs

The eigenvalues of the states with larger values of  $n$  can be accurately



estimated from the asymptotic formulae

$$Lt_{n \rightarrow \infty} \lambda_{2n} = \left( \frac{3\pi n}{2} \right)^{\frac{2}{3}} \left( 1 + \frac{1}{4n} \right)^{\frac{2}{3}} \quad (32)$$

$$Lt_{n \rightarrow \infty} \lambda_{2n+1} = \left( \frac{3\pi n}{2} \right)^{\frac{2}{3}} \left( 1 + \frac{3}{4n} \right)^{\frac{2}{3}} \quad (33)$$

which show that the sums  $S_1$  and  $S_2$  diverge. However if we look at the difference  $S$  between the sums over the inverses of the even and odd eigenvalues the terms in  $S$  have the asymptotic behaviour

$$Lt_{n \rightarrow \infty} ( \lambda_{2n}^{-1} - \lambda_{2n+1}^{-1} ) = \left( \frac{2}{3\pi n} \right)^{\frac{2}{3}} \frac{1}{3n} \quad (34)$$

which indicates that the integral over  $n$  would converge. The first 10 terms in the series for  $S$  are tabulated in the last column of Table 1 and add up to  $\sim 0.691$ . The remaining terms in the sum can be added by converting the sum over  $n$  into an integral over  $n$  and using the asymptotic estimate given by eq.(34) for the integrand. Such an evaluation gives

$$R = \int_n^\infty ( \lambda_{2n}^{-1} - \lambda_{2n+1}^{-1} ) dn = \frac{1}{2} \left( \frac{2}{3\pi n} \right)^{\frac{2}{3}} . \quad (35)$$

Hence an estimate of  $S$  is given by choosing  $n = 10.5$  in eq.(35) which gives the result that

$$S = \sum_{n=0}^{\infty} ( \lambda_{2n}^{-1} - \lambda_{2n+1}^{-1} ) \sim .691 + 0.037 = 0.728 \quad (36)$$

in good agreement with the exact result derivable from eq.(29) using the value  $\beta = \frac{1}{3}$  for  $N = 1$  which gives

$$S = \left( \frac{1}{3} \right)^{\frac{2}{3}} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})} \sim 0.729 . \quad (37)$$

By calculating more zeros of the Airy function and its derivative exactly the sum involving the eigenvalues can be evaluated more accurately without approximating part of the sum by an integral over  $n$  and the agreement between the two sides of eq.(29) can be verified to more significant figures.

The asymptotic limits given by eqs.(32) and (33) also indicate that even-though the sums over the inverses of the eigenvalues will not converge they also show that the sums over squares and higher powers of the inverse of the eigenvalues will converge. Hence higher order sum rules such as those arising from eq.(12) may be verified by using the appropriate eigenvalues in the sum and the appropriate Green's functions in the integral.

## 4.2 Simple Harmonic Oscillator $V(x) = x^2 + 1$

For the oscillator potential  $V(x) = x^2$  the eigenvalues are given by  $\lambda_n = 2n + 1$ . The sums  $S_1$  and  $S_2$  are both divergent but the difference  $S$  gives

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4} \quad (38)$$

in agreement with the value from eq.(29) for  $\beta = \frac{1}{4}$  corresponding to  $N = 2$ . The zero energy solutions to the Schroedinger equation have to be calculated numerically to verify the integral involving the Green's functions. However analytical results are possible for the oscillator shifted in energy by one unit. Therefore we consider this case in detail. For the shifted simple harmonic oscillator potential  $V(x) = x^2 + 1$  the eigenvalues are  $\lambda_n = 2n + 2$  with  $n$  taking positive even and odd values. The sums  $S_2$  and  $S_1$  over the inverses of the even and odd eigenvalues are both logarithmically divergent. However the difference between the sums gives

$$S = \sum_{n=0}^{\infty} \frac{(-)^n}{\lambda_n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots = \frac{\ln 2}{2} . \quad (39)$$

The zero energy solutions to the Schroedinger equation for this potential are gaussians and integrals over gaussians and they may be used to construct the Green's functions

$$G_1(x, x) = -\frac{2}{\sqrt{\pi}} e^{x^2} \int_0^x e^{-y^2} dy \int_{\infty}^x e^{-z^2} dz \quad (40)$$

$$G_2(x, x) = -e^{x^2} \int_{\infty}^x e^{-y^2} dy \quad (41)$$

$$G_2(x, x) - G_1(x, x) = \frac{2}{\sqrt{\pi}} e^{x^2} \int_{\infty}^x e^{-y^2} dy \int_{\infty}^x e^{-z^2} dz . \quad (42)$$

The sum rule  $S$  in this case gives rise to the relation

$$\sqrt{\pi} \int_0^\infty e^{x^2} (1 - \operatorname{erf}(x))^2 dx = \ln 2 \quad (43)$$

where  $\operatorname{erf}(x)$  is the error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy. \quad (44)$$

This is an interesting integral relation which has arisen from the sum rule for the shifted oscillator potential.

### 4.3 Quartic Potential $V(x) = x^4$

The low lying eigenvalues for the even states in the quartic potential can be obtained from Bender *et al* (1977) and the eigenvalues for the odd states can be extracted from Hioe and Montroll (1975) by appropriate renormalisation of the eigenvalues to accomodate the changed strength of the potential. The lowest eigenvalues are listed in Table 2.

$n$	$\lambda_{2n}$	$\lambda_{2n+1}$
0	1.060362	3.799673
1	7.455698	11.644746
2	16.261826	21.238373
3	26.528472	32.098598
4	37.923001	43.981158

Table 2: The low lying eigenvalues for the quartic potential

The eigenvalues for the higher lying states may be calculated using the WKB formula in eq.(22) for  $N = 4$  to give

$$\lambda_{2n} \sim \alpha \left( n + \frac{1}{4} \right)^{\frac{4}{3}} \quad (45)$$

$$\lambda_{2n+1} \sim \alpha \left( n + \frac{3}{4} \right)^{\frac{4}{3}} \quad (46)$$

where

$$\alpha = \left( 6\sqrt{\pi} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right)^{\frac{4}{3}}. \quad (47)$$

To verify the sum rules the first  $k$  terms may be summed exactly using the numerical estimates in Table 2 and the remainder may be evaluated by approximating by integrals over  $n$  and using the WKB estimates which give

$$\int_{k+\frac{1}{2}}^{\infty} \lambda_{2n}^{-1} dn = \frac{3}{\alpha} \left( k + \frac{3}{4} \right)^{-\frac{1}{3}} \quad (48)$$

$$\int_{k+\frac{1}{2}}^{\infty} \lambda_{2n+1}^{-1} dn = \frac{3}{\alpha} \left( k + \frac{5}{4} \right)^{-\frac{1}{3}}. \quad (49)$$

Setting  $k = 4$  we get

$$S_1 \sim 0.45003 + 0.31349 = 0.76352 \quad (50)$$

$$S_2 \sim 1.22266 + 0.30413 = 1.52679 \quad (51)$$

$$S \sim 0.77263 - 0.00936 = 0.76327 \quad (52)$$

to be compared with the values from eqs.(29)-(31) calculated using  $\beta = \frac{1}{6}$  corresponding to  $N = 4$  which give

$$S_1 = S = \frac{S_2}{2} = \left( \frac{1}{6} \right)^{\frac{4}{3}} \frac{\Gamma(\frac{1}{2}) \Gamma^2(\frac{1}{3})}{\Gamma(\frac{2}{3}) \Gamma(\frac{5}{6})} = 0.76330 \quad (53)$$

in good agreement with the numerical results. The agreement can be improved by calculating more of the eigenvalues exactly.

#### 4.4 Particle In a Box $N \rightarrow \infty$

To check the sum rules for the case  $N \rightarrow \infty$  we proceed as follows. The sum rules scale when the strength of the power law potential is changed by a factor  $\gamma$ . The scaling factor for the energy eigenvalue is easily found to be  $\gamma^{\frac{2}{N+2}}$ . Therefore if we consider the potential

$$\begin{aligned} Lt_{N \rightarrow \infty} \left( \frac{2|x|}{\pi} \right)^N &= 0 \text{ if } x < \frac{\pi}{2} \\ &= 1 \text{ if } x = \frac{\pi}{2} \\ &= \infty \text{ if } x > \frac{\pi}{2} \end{aligned} \quad (54)$$

which corresponds to a particle in a box with infinite potential at the walls at  $|x| = \frac{\pi}{2}$ . The eigenvalues for this potential are the squares of integers. Hence the sums over the inverses of the eigenvalues can all be found in closed form and are given by

$$\begin{aligned} S_1 &= \sum_{n=0}^{\infty} (2n+2)^{-2} = \frac{\pi^2}{24} \\ S_2 &= \sum_{n=0}^{\infty} (2n+1)^{-2} = \frac{\pi^2}{8} \\ S &= S_2 - S_1 = \frac{\pi^2}{12} . \end{aligned} \quad (55)$$

When the scaling factor is taken into account the sum rule in eq.(29) becomes

$$S = Lt_{\beta \rightarrow 0} \left( \frac{\pi\beta}{2} \right)^{2-4\beta} \frac{\Gamma(3\beta) \Gamma^2(2\beta)}{\Gamma(4\beta) \Gamma(1-\beta)} . \quad (56)$$

If the limiting behaviour of the gamma function  $Lt_{z \rightarrow 0} \sim z^{-1}$  is used then eqs.(56),(30) and (31) give

$$S = 2S_1 = \frac{2}{3}S_2 = \frac{\pi^2}{12} \quad (57)$$

in agreement with the results in eq.(55).

## 5 General Symmetric Confining Potentials

In this section we consider general symmetric confining potentials and examine the structure of the sum rules. Let  $\xi_1$  and  $\xi_2$  be two linearly independent solutions of the zero energy Schroedinger equation

$$\frac{d^2}{dx^2} \Psi(x) = V(x) \Psi(x) \quad (58)$$

such that

$$Lt_{x \rightarrow 0} \xi_1 = a , \quad Lt_{x \rightarrow 0} \xi_2 = b x . \quad (59)$$

Then a solution  $\Phi_2$  which satisfies  $Lt_{x \rightarrow \infty} \Phi_2 \rightarrow 0$  may be constructed as the linear combination

$$\Phi_2 = \xi_1 + c \xi_2 \quad (60)$$

and the solution  $\Phi_1$  may be chosen as either  $\xi_1$  or  $\xi_2$  to construct the Green's functions using eqs.(8)-(10). Then

$$G_1(x, y) = + \frac{1}{ab} \xi_2(x_{<}) \Phi_2(x_{>}) \quad (61)$$

$$G_2(x, y) = - \frac{1}{abc} \xi_1(x_{<}) \Phi_2(x_{>}) \quad (62)$$

$$G_2(x, y) - G_1(x, y) = - \frac{1}{abc} \Phi_2(x_{<}) \Phi_2(x_{>}) \quad (63)$$

are the Green's functions corresponding to the odd and even eigenstates and the difference between them. These Green's functions may be used in eq.(11) to establish the sum rules

$$S_1 = \sum_{n=0}^{\infty} \frac{1}{\lambda_{2n+1}} = \frac{1}{ab} \int_0^{\infty} \xi_2(x) \Phi_2(x) dx \quad (64)$$

$$S_2 = \sum_{n=0}^{\infty} \frac{1}{\lambda_{2n}} = - \frac{1}{abc} \int_0^{\infty} \xi_1(x) \Phi_2(x) dx \quad (65)$$

$$S = \sum_{n=0}^{\infty} \frac{(-)^n}{\lambda_n} = - \frac{1}{abc} \int_0^{\infty} \Phi_2^2(x) dx. \quad (66)$$

By noting that the boundary conditions satisfied by the functions specified in eqs.(57) and (58) show that

$$\begin{aligned} Lt_{x \rightarrow 0} \frac{d}{dx} \Phi_2^2 &= 2abc \\ Lt_{x \rightarrow 0} \frac{d}{dx} (\xi_1 \Phi_2) &= abc \\ Lt_{x \rightarrow 0} \frac{d}{dx} (\xi_2 \Phi_2) &= ab \end{aligned} \quad (67)$$

all three sum rules may be written in the compact form

$$\left( \frac{d^2 f}{dx^2} \right) |_{x=0} = \frac{1}{\Delta} f|_{x=0} \quad (68)$$

with three different choices of  $\Delta$  and  $f(x)$  as given by

$$\begin{aligned}\Delta = -S_1, \quad f(x) &= \int_{-\infty}^x \xi_2(x) \Phi_2(x) dx \\ \Delta = +S_2, \quad f(x) &= \int_{-\infty}^x \xi_1(x) \Phi_2(x) dx \\ \Delta = +\frac{S}{2}, \quad f(x) &= \int_{-\infty}^x \Phi_2^2(x) dx.\end{aligned}\tag{69}$$

If we restore  $\hbar$  and the mass parameter of the system then eq.(68) has the appearance of a Schroedinger like equation for the incomplete overlap integrals  $f(x)$  evaluated at  $x = 0$  with the sums over the inverses of the eigenvalues playing the role of the inverse of an energy-like parameter. We have shown that the sum over the difference between the inverses of the even and odd eigenvalues in a symmetric confining potential can be calculated from the integral of the zero energy solution of the Schroedinger equation which is a decaying solution in the asymptotic region.

In this paper it has been shown that for symmetric confining potentials it is possible to establish sum rules involving the eigenvalues of odd and even states separately. It has been shown that even when the sum rules for even and odd states do not converge the difference between them could converge. We have shown that for power law potentials simple expressions for the sum rules may be derived which show an interesting structure. Eventhough the eigenvalues for  $N \neq 2$  have to be found numerically the sum rules show that the inverses of the eigenvalues add up to some simple ratios involving the well known gamma functions. This is an interesting result from a mathematical point of view. Higher order sum rules such as those emerging from the application of eq.(12) indicate even more intricate relationships between the solutions of the Schroedinger equation.

## 6 References

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